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ON THE NUMBER OF LINES IN PLANAR SPACES

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Suppose $\mathbf{S}$ is a planar space with $v > 4$ points and let q be the positive real number such that $v = q^3 + q^2 + q + 1$. Assuming a weak non-degeneracy condition, we shall show that $\mathbf{S}$ has at least $(q^2+1)(q^2+q+1)$ lines with equality iff q is a prime power and $\mathbf{S}=\mathrm{PG}(3,q)$.

1. Introduction

A *linear space* is a pair $\mathbf{S}=(\mathcal{P},\mathcal{L})$ consisting of a set $\mathcal{P}$ of *points* and a set $\mathcal{L}$ of subsets of $\mathcal{P}$, called *lines*, such that any two distinct points occur in a unique line, every line has at least two points, and there are at least two lines. A *subspace* of $\mathbf{S}$ is a set S of points such that every line that has two points in S is contained in S. A *planar space* is a linear space together with a set $\mathcal{E}$ of subspaces, called *planes*, such that any three non-collinear points occur in a unique plane, every plane has three non-collinear points, and there are at least two planes.

The famous de Bruijn–Erdős–Hanani Theorem ([1,4]) says that every linear space has at least as many lines as points with equality iff it is a (possibly degenerate) projective plane. In this paper we prove an analogous result for planar spaces.

Let $\mathbf{S}$ be a planar space, v the number of its points and b the number of its lines. Dowling and Wilson ([2]) showed that $b \geq 2(v-1)$ with equality iff $\mathbf{S}$ is the direct product of a (possibly degenerate) projective plane and a point, which means that $\mathbf{S}$ has a plane with $v-1$ points that is a projective plane. A planar space with a plane E containing "almost" all points is degenerate in the sense that its structure is dominated by the structure of E. Assuming a weak non-degeneracy condition, one can improve the bound for the number of lines substantially.

Theorem 1.1. *Let $\mathbf{S}$ be a finite planar space with $v > 4$ points and b lines, and let q be the unique positive real number satisfying $v = q^3 + q^2 + q + 1$. If every plane has*

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at most q^3+q^2+1 *points, then* $b \geq (q^2+1)(q^2+q+1)$ *with equality iff* q *is a prime power and* $\mathbf{S}$ *is the 3-dimensional projective space* $PG(3,q)$ *of order* q.

Remark. If some plane E has $v-c$ points, $v=q^3+q^2+q+1$, $c<q$, then $\mathbf{S}$ has at least $(c+1)(v-c)-\binom{c}{2}$ lines (at least $|E|=v-c$ lines in E and at least $c(v-c)-\binom{c}{2}$ lines joining the c points not on E to the points on E). Planar spaces with $b=(c+1)(v-c)-\binom{c}{2}$ can be constructed as follows.

Consider a partition $M_1,\dots,M_s$ of the set M of unordered pairs $\{i,j\}$, $i,j=1,\dots,c$ and $i \neq j$, with the property that all pairs contained in the same set M_k are disjoint. Choose a projective plane with $v-c$ points which has a line l with at least s points and choose points $Q_1,\dots,Q_s$ on l. For example, E can be a degenerate projective plane, which has a line l of size $v-c-1 \geq \binom{c}{2}$, $s=\binom{c}{2}$, and $M_1,\dots,M_s$ is the trivial partition of M. The planar space consists of the plane E and c points $P_1,\dots,P_c$ not on E and such that the line joining P_i and P_j contains the point Q_k where $\{i,j\}$ is in M_k.

One might also start with a 3-dimensional projective space $\mathbf{P}$, a plane E of $\mathbf{P}$ and c points $P_1,\dots,P_c$ not on E no three of which are collinear. Then let $\mathbf{S}$ be the planar space induced by $\mathbf{P}$ on $E \cup \{P_1,\dots,P_c\}$.

2. Proof of the theorem

In this section, $\mathbf{S}$ denotes a planar space with $v>4$ points and b lines. The number of lines on a point P is denoted by r_P and is called the *degree* of P. The number of points on a line l is denoted by k_l and is called the *degree* of l.

We define a function f by $f(s,v)=s^2(v-s)/(v-1)$ for $2 \leq s \leq v-1$. The following lemma is a variant of a result of Stanton and Kalbfleisch [5].

Lemma 2.1. *Suppose that* S *is a subspace of* $\mathbf{S}$ *with* s *points. Then there are at least* $f(s,v)$ *lines which intersect* S *in precisely one point.*

Proof. Denote by $\mathcal{M}$ the set of lines that meet S in precisely one point. Since every point of S is joined to every point outside of S by a line of $\mathcal{M}$, we have $\sum_{l\in\mathcal{M}}(k_l-1)=s(v-s)$. Since the lines of $\mathcal{M}$ cover every pair of points outside of S at most once, we have $\sum_{l\in\mathcal{M}}(k_l-1)(k_l-2) \leq (v-s)(v-s-1)$. It follows that $\sum_{l\in\mathcal{M}}(k_l-1)^2 \leq (v-s)(v-1)$. Using the inequality of Cauchy and Schwarz, we conclude that

$$s^2(v-s)^2=\left(\sum_{l\in\mathcal{M}}(k_l-1)\right)^2 \leq |\mathcal{M}|\sum_{l\in\mathcal{M}}(k_l-1)^2 \leq |\mathcal{M}|(v-s)(v-1).$$

Consequently $|\mathcal{M}| \geq f(s,v)$. ∎

Let q be the unique positive real number such that the number of points is $v=q^3+q^2+q+1$. We suppose that every plane has at most q^3+q^2+1 points and that the number of lines is at most $(q^2+1)(q^2+q+1)$. We shall show that q is a prime power and $\mathbf{S}=PG(3,q)$.

Lemma 2.2. *Every plane has at most q^2+q+1 points.*

Proof. Let E be a plane and set $e=|E|$. Since E contains at least e lines (by the de Bruijn–Erdős–Hanani Theorem mentioned in the introduction), Lemma 2.1 shows that $b \ge e+f(e,v)=:g(e,v)$.

Assume that $q^2+q+1<e\le q^3+q^2+1$. Since the function $x\to g(x,v)$ has two extrema, a minimum for a negative value of x and a maximum for a positive value of x, it follows that $b>g(q^2+q+1,v)$ or $b\ge g(q^3+q^2+1,v)$. Since $b\le (q^2+1)(q^2+q+1)$, both cases lead to a contradiction. ∎

Lemma 2.3. *If the average point degree is at least q^2+q+1, then* $\mathbf{S}=PG(3,q)$.

Proof. Denote the average point degree by r and suppose that $r\ge q^2+q+1$. We have $\sum_{l\in\mathcal{L}} k_l(k_l-1)=v(v-1)$ and $\sum_{l\in\mathcal{L}} k_l=\sum_{P\in\mathcal{P}} r_P=v\cdot r$. Hence $\sum_{l\in\mathcal{L}} k_l^2 = v(v-1+r)=v(q^3+q^2+q+r)\le v(q\cdot r+r)=v(q+1)r$. Denote by b the number of lines. Then, using the inequality of Cauchy and Schwarz, we obtain

$$bv(q+1)r \ge b\sum_{l\in\mathcal{L}} k_l^2 \ge \left(\sum_{l\in\mathcal{L}} k_l\right)^2 = v^2r^2. \tag{1}$$

Consequently $b\ge \frac{vr}{q+1}=(q^2+1)r\ge (q^2+1)(q^2+q+1)$. Hence, $b=(q^2+1)(q^2+q+1)$ and equality holds in (1), which implies that every line has the same degree k. It is given by $bk(k-1)=v(v-1)$, so $k=q+1$ and every plane is a 2-design with line degree $q+1$. Since planes have at most q^2+q+1 points, it follows that every plane is a projective plane of order q. The points are on $\frac{v-1}{k-1}=q^2+q+1$ lines and each line lies in $\frac{v-k}{q^2}=q+1$ planes. Hence, q must be a prime power and $\mathbf{S}=PG(3,q)$. ∎

Lemma 2.4. *A line of degree k meets at least $\frac{k^2(v-k)}{q^2+q}$ other lines.*

Proof. Suppose that a E is a plane that contains the line l of degree k. Then, by Lemma 2.1, l meets at least $\frac{k^2(|E|-k)}{|E|-1}\ge \frac{k^2(|E|-k)}{q^2+q}$ lines of E. Taking the sum over the planes containing l, we see that l meets at least $\frac{k^2(v-k)}{q^2+q}$ lines. ∎

Lemma 2.5. *Suppose that $k_1,\dots,k_r$, $r\ge 1$, are positive integers satisfying $3(k_i+k_j)\le 2v$ for different indices $i,j\in\{1,\dots,r\}$. Set $k:=\frac{1}{r}\sum_{i=1}^r k_i$ and define $f(x):=x^2(v-x)$. Then $\sum_{i=1}^r f(k_i)\ge rf(k)$.*

Proof. We proceed by induction on the number n of indices i satisfying $k_i\ne k$. If $n=0$, then the statement is trivial. Suppose that $n>0$. Then there are indices i and j with $k_i<k<k_j$. We may assume that $k_1<k<k_2$. Put $d:=k_2-k$. Then

$$f(k_1)+f(k_2)-f(k_1+d)-f(k_2-d)=d(2v-3k_1-3k_2)(k_2-k_1-d)\ge 0,$$

since $k_2-k_1-d=k-k_1>0$. It suffices therefore to verify the statement, if we replace k_1 by k_1+d and k_2 by $k_2-d=k$. But this follows from the induction hypothesis, since still $k=\frac{1}{r}\sum_{i=1}^r k_i$ and $3(k_i+k_j)\le 2v$ for different indices i and j

(this is trivial, if $i,j \neq 1$ or $\{i,j\}=\{1,2\}$; also $3(k_1+k_i) \leq 2v$ for $i \geq 3$, since the new value of k_1 is still less then the old value of k_2). ∎

Lemma 2.6. *If $\mathcal{M}$ is the set of lines passing through a point P, then $\sum_{l \in \mathcal{M}} k_l^2(v-k_l) \geq r_P k^2(v-k)$ where k is the average degree of the lines in $\mathcal{M}$.*

Proof. This will follow from the preceding lemma, if we can show that $3(k_l+k_h) \leq 2v$ for any distinct lines $l,h \in \mathcal{M}$. Consider lines $l,h \in \mathcal{M}$. The sum of their degrees is at most one more than the number of points in the plane they span, so $k_h+k_l \leq q^2+q+2$ and hence $k_h+k_l \leq \lfloor q^2+q+2 \rfloor$.

Assume that $2v \leq 3(k_l+k_h)$. Then $2(q^3+q^2+q+1) \leq 3(q^2+q+2)$, which implies that $q<2$. Hence $2v \leq 3(k_l+k_h) \leq 3 \cdot \lfloor q^2+q+2 \rfloor \leq 3 \cdot 7 = 21$, and thus $v \leq 10$.

If $v=q^3+q^2+q+1=10$ then $q<1.7$; therefore $q^2+q+2<7$ and hence $2v> 3 \cdot \lfloor q^2+q+2 \rfloor$, a contradiction.

Next consider the case in which $v=9$. Then $q<1.6$, so $q^2+q+1<6$ and the number of lines is $b \leq (q^2+1)(q^2+q+1)<19$. By Lemma 2.2, every plane has at most 5 points. On the other hand, $18=2v \leq 3(k_l+k_h)$, so $k_l+k_h \geq 6$. Hence $k_l+k_h= 6$ and the plane E spanned by h and l has 5 points. If l has 4 points, then every other line meeting l has 2 points and there must be $4(v-4)=20$ of them. But $b< 19$, so l and h have degree 3 and E has 6 lines. Lemma 2.1 implies that there are at least $5^2(v-5)/(v-1)>12$ lines that meet E in one point. Hence $b>6+12=18$, a contradiction.

If $v=8$ then $q<1.5$ and $q^2+q+2<6$. But $2v \leq 3 \cdot \lfloor q^2+q+2 \rfloor$, so $v=8$ is not possible.

Assume that $v=7$. Then $q<1.4$, $q^2+q+1<5$ and $b \leq (q^2+1)(q^2+q+1)<13$. Hence every plane has at most 4 points, which implies that the lines have size 2 or 3 and distinct lines of size 3 are disjoint. Thus there are at most two 3-lines. It follows that the number of 2-lines is at least $\frac{1}{2}[v(v-1)-2\cdot 3\cdot 2]=15$, a contradiction.

If $v \in \{5,6\}$, then $q<1.3$, so planes have at most $q^2+q+1<4$ points, which implies that every line has degree 2; therefore $b=\frac{1}{2}v(v-1)$, a contradiction, since $b \leq (q^2+1)(q^2+q+1)$. ∎

Lemma 2.7. *The number q is a prime power and $\mathbf{S}=PG(3,q)$.*

Proof. In view of Lemma 2.3, we may assume that the average point degree is at most q^2+q+1. We shall show that in this case every point has degree q^2+q+1. Let P_0 be a point and denote by r its degree and by $\mathcal{M}$ the set of lines on P_0. If $l \in \mathcal{M}$, then Lemma 2.4 shows

$$\sum_{P \in l \setminus \{P_0\}} (r_P - 1) \geq \frac{k_l^2(v-k_l)}{q^2+q} - (r-1).$$

Consequently

$$\sum_{P \in \mathcal{P}} (r_P - 1) \geq (r-1) + \sum_{l \in \mathcal{M}} \left(\frac{k_l^2(v-k_l)}{q^2+q} - (r-1) \right).$$

Since the average point degree is at most q^2+q+1, it follows that

$$(r-1)^2+v(q^2+q) \geq \sum_{l \in \mathcal{M}} \frac{k_l^2(v-k_l)}{q^2+q}. \tag{2}$$

The average degree k of the lines of $\mathcal{M}$ is given by $v-1=r(k-1)$. Therefore (2) and the preceding lemma imply that

$$(r-1)^2+v(q^2+q) \geq r\frac{k^2(v-k)}{q^2+q}.$$

Assume that $r \leq q^2+q+1$. Then $k \geq q+1$, since $v-1=r(k-1)$. Also $k \leq q^2+q \leq v-q-1$, since every plane has at most q^2+q+1 points. Since the polynomial f defined by $f(x):=x(v-x)$ has degree 2 and negative leading coefficient, and in view of $f(q+1)=f(v-q-1)$, it follows that $k(v-k)=f(k) \geq f(q+1)=(q+1)(v-q-1)=q^2(q+1)^2$. Hence

$$(r-1)^2+v(q^2+q) \geq rkq(q+1)=(v-1+r)q(q+1).$$

Consequently $(r-1)^2 \geq (r-1)q(q+1)$, so $r \geq q^2+q+1$. Since P_0 was any point, it follows that the average point degree is at least q^2+q+1. Lemma 2.3 completes the proof of the lemma. ∎

A problem

Can a theorem similar to Theorem 1.1 be proved for the number of lines in a geometric lattice of any dimension? More generally, for $q>1$ and $d \geq 2$, define

$$\Theta_d := \frac{q^{d+1}-1}{q-1} = q^d+\ldots+q+1 \quad \text{and} \quad \Psi_{s+1}^{d+1} := \prod_{i=0}^{s} \frac{\Theta_{d-i}}{\Theta_i}.$$

Assuming some non-degeneracy condition, is it possible to prove that a geometric lattice of rank $d+1$ (hence dimension d) with Θ_d points has at least Ψ_{s+1}^{d+1} subspaces of rank $s+1$ (dimension s), $1 \leq s \leq d-1$, with equality iff it is $PG(d,q)$? This is known to be true only if $s=d-1$ (see [3]) or $d=3$ and $s=1$ (Theorem 1.1).

References

[1] N. G. de Bruijn, and P. Erdős: On a combinatorial problem, *Indag. Math.* **10** (1948), 421–423.

[2] T. A. Dowling, and M. Wilson: The slimmest geometric lattices, *Trans. Am. Math. Soc.* **196** (1974), 203–215.

[3] C. Greene: A rank inequality for finite geometric lattices, *J. Comb. Th.* **9** (1970), 357–364.

[4] H. HANANI: On the number of straight lines determined by n points, *Riveon Lematematika* **5** (1951), 10–11.

[5] R. G. STANTON, and J. G. KALBFLEISCH: The $\lambda-\mu$ problem: $\lambda=1$ and $\mu=3$, in: *Proc. Second Chapel Hill Conf. on Combinatorics, Chapel Hill*, 451–462, 1972.

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